A General Model for Partially Reliable Information Sources

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Abstract – *Combining testimonial reports from independent and partially reliable sources is an important problem of uncertain reasoning. This paper proposes a general model of partially reliable sources which includes several previously known results as special cases. The paper reproduces these results, gives a number of new insights, and thereby contributes to a better understanding of this important AI application.*[1]

Keywords: Evidence Theory, Belief Functions, Combining Testimonial Reports, Unreliable Sources, Sensor Fusion.

1 INTRODUCTION

An important application of uncertain reasoning is the problem of combining reports or data from partially reliable witnesses, experts, sensors, or measurement instruments. In legal cases, for example, testimonies from different witnesses must be taken into account. Due to the relative unreliability of witnesses, testimonies do typically not fully confirm each other. Very often, testimonies are even totally conflicting. Judges or jurors must then resolve such conflicts and and pass their sentence accordingly. Similarly, if several reports from sensors or data obtained from different measurement instruments are given, it is typical to have conflicts or inconsistencies. This is again due to the relative unreliability of sensors or measurement instruments.

The common feature of situations like the ones mentioned above is that partly or fully *conflicting information* is obtained from *partially reliable sources*. The question of how to combine such conflicting information is a very general problem of uncertain reasoning. We will refer to it as the *problem of partially reliable sources*.

There are numerous ways to attack the problem of partially reliable sources. In order to simplify matters, it is usually assumed that

- there are only two alternative hypotheses hyp and $\neg hyp$ (e.g. suspect X is guilty vs. suspect X is innocent),
- the sources are *identical* in the sense that they are characterized by the same parameters,
- the sources are conditional *independent* given the hypothesis (the hypothesis is the only common parameter that determines the outcomes of their reports).

These assumptions are somehow idealistic, but one can imagine practical situations in which they are approximately correct. We will accept these restrictions throughout this paper and postpone more general models to future publications. Because only two alternative hypotheses hyp and $\neg hyp$ are considered, we can distinguish between *positive reports* and *negative reports*. In general, we consider n positive and m negative reports. $N = n + m$ is the total number of reports.

The first way to deal with partially reliable sources is to assume that there is a certain chance $p = p(rep_i|hyp)$ that source i yields a positive report rep_i in case of hyp. On the other hand, there is certain chance $q = p(rep_i|\neg hyp)$ that the source yields a positive report rep_i in case of $\neg hyp$. q represents the ratio of *false positives* and $\bar{p} = 1 - p$ the ratio of *false negatives*. These values are well documented for many medical tests.

If N independent information sources of this type provide reports, we get a simple Bayesian network [1, 2] in which root node HYP precedes N report nodes REP_1 to REP_N. The possible values of HYP are hyp and $\neg hyp$. Similarly, the possible values of REP_i are rep_i and $\neg rep_i$. Let $h = p(hyp)$ and $\bar{h} = 1 - h$ denote the prior probabilities of hyp and $\neg hyp$, respectively. If the first n reports are all positive and the remaining $m = N - n$ reports are negative, then we get the following posterior probability for hyp:

$$p^*(hyp) = p(hyp|rep_1, \ldots, rep_n, \neg rep_{n+1}, \ldots, \neg rep_N) = \frac{h}{h + \bar{h}\left(\frac{q}{p}\right)^n\left(\frac{\bar{q}}{\bar{p}}\right)^m}. \quad (1)$$

For a detailed discussion and interpretation of this formula we refer to [3]. In Subsection 3.3 we will reproduce the same result as a special case from a more general model.

Another approach is to model partially reliable sources on the basis of *reliability variables* REL_i with rel_i and $\neg rel_i$ as possible values [3]. It is then assumed that the report of a reliable source is positive whenever the hypothesis is true and negative whenever the hypothesis is

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false. More formally, we have $p(rep_i|hyp, rel_i) = 1$ and $p(rep_i|\neg hyp, rel_i) = 0$. On the other hand, if source i is unreliable, then the outcome of REP_i only depends on a randomization parameter a_i no matter what the state of HYP is. In other words, we have $p(rep_i|hyp, \neg rel_i) = p(rep_i|\neg hyp, \neg rel_i) = a_i$. This leads to a Bayesian network in which H and REL_i are the parents of REP_i for all $1 \leq i \leq N$. Consider again n positive and m negative reports. If we assume the same prior probability $p(rel_i) = \rho$ and the same randomization parameter $a_i = a$ for all $1 \leq i \leq N$, we get the following posterior probability for hyp:

$$\begin{aligned} p^*(hyp) &= p(hyp|rep_1, \ldots, rep_n, \neg rep_{n+1}, \ldots, \neg rep_N) \\ &= \frac{h}{h + \bar{h}\big(\frac{a\bar{\rho}}{1-\bar{a}\bar{\rho}}\big)^n \big(\frac{1-a\bar{\rho}}{\bar{a}\bar{\rho}}\big)^m} . \end{aligned} \tag{2}$$

Again, we refer to [3] for a detailed discussion of this result and its relationship to (1). We will also reproduce the this formula in Subsection 3.4 as a special case of the general model.

A third possible model of partially reliable sources results from looking at source i as somebody who is either telling the truth or purposely lying due to some lack of veracity. Let ρ and $\bar{\rho} = 1 - \rho$ be the corresponding probabilities. Furthermore, suppose that all N information sources provide positive reports. This means that they are all either telling the truth with probability ρ^N or lying with probability $\bar{\rho}^N$. In the first case, hyp must be true, whereas in the second case hyp must be false. This leads to

$$p^*(hyp) = p(hyp|rep_1, \ldots, rep_N) = \frac{\rho^N}{\rho^N + \bar{\rho}^N} . \tag{3}$$

This formula goes back to Laplace (see Chapt. XI of [4]) and is related to the *Condorcet Jury Theorem* discussed in social choice theory [5]. It is remarkable that it does not depend on a prior distribution over HYP.[2] Boole gives a similar formula that includes a prior distribution over HYP (see Chapt. XXI of [6]). He considers the isomorphic problem of n independent jurors, rendering an unanimous verdict, being correct. In Subsection 3.2 we will reproduce this formula for the case of n positive and m negative reports.

The goal of this paper is to define a general model of partially reliable sources from which results like the ones mentioned above drop out as special cases. We choose *Dempster-Shafer theory of evidence* (DS-theory) as the underlying mathematical theory of uncertainty [7, 8]. By looking at DS-theory from the perspective of the *theory of hints* [9], that is by interpreting masses as probabilities of interpretations, we will get a very clear picture of how the general model relates to more specific models such as the ones introduced above. Furthermore, Dempster's rule of combination turns out to be the key mechanism to solve the problem of partially reliable sources. We will also see that prior knowledge about HYP, if available, is an important parameter. But interesting results are also obtained for the case where no prior knowledge is available. In fact, prior knowledge will be considered as a continuous quantity with "no prior knowledge" and $p(hyp) = h$ as special cases.

[2] This is due to the particularity of this model, in which, for a given set of reports, the true state of the variable HYP is unambiguously determined by the corresponding values of the variables REP_i. In other words, the incoming set of reports extends the range of the prior distribution from $\mathbf{P}(REL_1, \ldots, REL_n)$ to $\mathbf{P}(HYP, REL_1, \ldots, REL_n)$ and allows then to compute posterior probabilities for hyp.

2 THE GENERAL MODEL

From an abstract point of view, we can say that every information source i delivers some information $\mathcal{I}_i$ about the set of variables $d(\mathcal{I}_i) = \{HYP, REP_i\}$ called *domain* of $\mathcal{I}_i$. The only common variable is HYP, the one we are interested in. This reflects our assumption of independent sources. As a consequence, before combining the various pieces of information $\mathcal{I}_i$, it is possible to focus them to the common domain $\{HYP\}$. In other words, every information source i delivers some information $\mathcal{H}_i = \mathcal{I}_i^{\downarrow\{HYP\}}$ about HYP. Finally, if $\mathcal{H}_0$ represents some prior knowledge about HYP, the problem is to compute the combined information

$$\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N , \tag{4}$$

and to use $\mathcal{H}$ in order to draw conclusions about HYP.

Now let every information $\mathcal{H}_i$ be represented by a *mass function* m_i in the sense of DS-theory [7, 8]. By $m = m_0 \otimes \cdots \otimes m_N$ we denote the combined mass function obtained as a result of applying Dempster's rule of combination $\otimes$. Because HYP is a binary variable, there are four possible subsets $\emptyset$, $\{hyp\}$, $\{\neg hyp\}$, and $\{hyp, \neg hyp\}$ to be considered. Without loss of generality, we assume that

$$\begin{aligned} m_i(\emptyset) &= 0, \\ m_i(\{hyp\}) &= \bar{x}_i = 1 - x_i, \\ m_i(\{\neg hyp\}) &= \bar{y}_i = 1 - y_i, \\ m_i(\{hyp, \neg hyp\}) &= z_i = 1 - \bar{x}_i - \bar{y}_i = x_i + y_i - 1, \end{aligned}$$

are the individual masses of these subsets. Note that cases of $m_i(\emptyset) \neq 0$ are avoided by normalization. At this point, we do not try to interpret the parameters x_i, y_i, and z_i. We will see in Subsection 2.1 and 2.2 that such an interpretation depends crucially on the choice of the concrete model. The idea here to generate general results from using DS-theory as the underlying mathematical mechanism. The interpretation of the results will then become clear when concrete models are considered.

Finally, with respect to the combined mass function m representing the aggregated information from all sources, we are interested in *degrees of support*

$$dsp(\{hyp\}) = \frac{m(\{hyp\})}{1 - m(\emptyset)} \tag{5}$$

and *degrees of possibility* $dps(\{hyp\}) = 1 - dsp(\neg hyp)$.[3] Furthermore, we interpret $dps(\{hyp\}) - dsp(\{hyp\})$ as a quantitative measure of ignorance [11].

[3] In accordance with [10, 12, 9], we prefer to speak about degree of support and degree of possibility of H instead of belief $Bel(H)$ and plausibility $Pl(H)$, respectively.

Theorem 1 *If m_0 to m_N are mass functions over HYP as defined above, then degree of support and degree of possibility with respect to the combined mass function m are determined by*

$$dsp(\{hyp\}) = \frac{\prod_{i=0}^{N} y_i - \prod_{i=0}^{N} z_i}{\prod_{i=0}^{N} x_i + \prod_{i=0}^{N} y_i - \prod_{i=0}^{N} z_i}, \quad (6)$$

and

$$dps(\{hyp\}) = \frac{\prod_{i=0}^{N} y_i}{\prod_{i=0}^{N} x_i + \prod_{i=0}^{N} y_i - \prod_{i=0}^{N} z_i}, \quad (7)$$

respectively.

Proof: We can use the fact that Dempster's rule of combination corresponds to multiplication of corresponding *commonality* functions q_i defined by $q_i(H) = \sum\{m_i(A) : A \supseteq H\}$ [7]. In our concrete case, we have $q_i(\emptyset) = 1$, $q_i(\{hyp\}) = \bar{x}_i + z_i = y_i$, $q_i(\{\neg hyp\}) = \bar{y}_i + z_i = x_i$, and $q_i(\{hyp, \neg hyp\}) = z_i$. As a consequence, we get $q(\emptyset) = 1$, $q(\{hyp\}) = \prod_{i=0}^{N} y_i$, $q(\{\neg hyp\}) = \prod_{i=0}^{N} x_i$, and $q(\{hyp, \neg hyp\}) = \prod_{i=0}^{N} z_i$. This leads to $m(\emptyset) = 1 - \prod_{i=0}^{N} x_i - \prod_{i=0}^{N} y_i + \prod_{i=0}^{N} z_i$, $m(\{hyp\}) = \prod_{i=0}^{N} y_i - \prod_{i=0}^{N} z_i$, $m(\{\neg hyp\}) = \prod_{i=0}^{N} x_i - \prod_{i=0}^{N} z_i$, and $m(\{hyp, \neg hyp\}) = \prod_{i=0}^{N} z_i$. Finally, (6) results from applying the normalization constant $\frac{1}{1-m(\emptyset)}$ to $m(\{hyp\})$. Similarly, (7) is obtained from $m(\{\neg hyp)$ and $dps(\{hyp\}) = 1 - dps(\{\neg hyp\})$.
□

Now let there be n *identical* positive and $m = N - n$ *identical* negative reports. Suppose that the positive reports m_1 to m_n are determined by corresponding parameters x_1, y_1, and z_1. Similarly, let x_2, y_2, and z_2 be the parameters of the negative reports m_{n+1} to m_N. Finally, if x, y, and z are the parameters of the prior knowledge m_0, we can transform the results of Theorem 1 into

$$\begin{aligned} dsp(\{hyp\}) &= \frac{y\, y_1^n y_2^m - z\, z_1^n z_2^m}{x\, x_1^n x_2^m + y\, y_1^n y_2^m - z\, z_1^n z_2^m} \\ &= 1 - \frac{x\, x_1^n x_2^m}{x\, x_1^n x_2^m + y\, y_1^n y_2^m - z\, z_1^n z_2^m}, \quad (8) \end{aligned}$$

$$dps(\{hyp\}) = \frac{y\, y_1^n y_2^m}{x\, x_1^n x_2^m + y\, y_1^n y_2^m - z\, z_1^n z_2^m}. \quad (9)$$

These two formulas can be regarded as the key to a solution of the problem of partially reliable sources. At this point, we do not further comment these results by interpreting the parameters involved. But we will introduce concrete models of partially reliable sources in the following subsection and illustrate corresponding applications of the above formulas in Section 3.

Let's have a closer look at some special cases. First, consider the case where a prior distribution $p(hyp) = h$ is given. This means that $x = \bar{y} = \bar{h} = 1 - h$ and $z = 0$ are the parameters of the prior knowledge m_0. Note that $z = 0$ implies $dsp(\{hyp\}) = dps(\{hyp\})$. The expressions in (8) and (9) can then be transformed into

$$dsp(\{hyp\}) = dps(\{hyp\}) = \frac{h}{h + \bar{h}\left(\frac{x_1}{y_1}\right)^n \left(\frac{x_2}{y_2}\right)^m}. \quad (10)$$

Furthermore, consider the cases where the prior knowledge is completely prejudiced by $h = 0$ or $h = 1$. Then the above formula simplifies to $dsp(\{hyp\}) = dps(\{hyp\}) = 0$ and $dsp(\{hyp\}) = dps(\{hyp\}) = 1$, respectively. Such a definite prejudice is therefore not compensable even if $n-m$ or $m-n$ tends to infinity. Other interesting cases arise from $x_1 = y_1$ or $x_2 = y_2$. This means that either the positive or the negative reports are valueless. Note how the corresponding expressions drop out of the denominator of (10). If all reports are valueless, that is if simultaneously $x_1 = y_1$ and $x_2 = y_2$, then $dsp(\{hyp\}) = dps(\{hyp\}) = h$, as one would expect.

Second, consider the case of $z = 1$. This is a situation where no prior knowledge is available. It implies $x = 1$ and $y = 1$. As a consequence, we get

$$dsp(\{hyp\}) = 1 - \frac{x_1^n x_2^m}{x_1^n x_2^m + y_1^n y_2^m - z_1^n z_2^m}, \quad (11)$$

$$dps(\{hyp\}) = \frac{y_1^n y_2^m}{x_1^n x_2^m + y_1^n y_2^m - z_1^n z_2^m}. \quad (12)$$

Note that the same result is obtained by omitting the prior knowledge m_0 from the beginning.

In general, we have $0 < z < 1$. We will discuss in Subsection 2.3 how to interpret this general situation. Prior knowledge will then turn out to be a continuous quantity in which z determines its strength and with "no prior knowledge" and $p(hyp) = h$ as opposite extreme cases.

2.1 MODELING PARTIALLY RELIABLE SOURCES

So far, we have discussed the general case in which prior knowledge, positive reports, and negative reports are all determined by corresponding parameters. The question now is how to derive these parameters from concrete models of partially reliable sources such as the ones introduced in Section 1. Before doing so, we will first identify a number of various models and discuss their relationship. We use an assumption-based propositional language to describe the models [10, 12]. Figure 1 gives an overview.

(T) *Truth Teller*: In this simple model we assume that there is a certain chance $p(rel) = \rho$ that the information source tells the truth and the report is correct. If this is all we know, we can express our knowledge by

$$rel \rightarrow (hyp \leftrightarrow rep).$$

Note that nothing is said about the behavior of the information source in case of $\neg rel$.

(L) *Liar*: This model is symmetric to the first one. We assume that there is a certain chance $p(\neg rel) = \bar{\rho}$ that

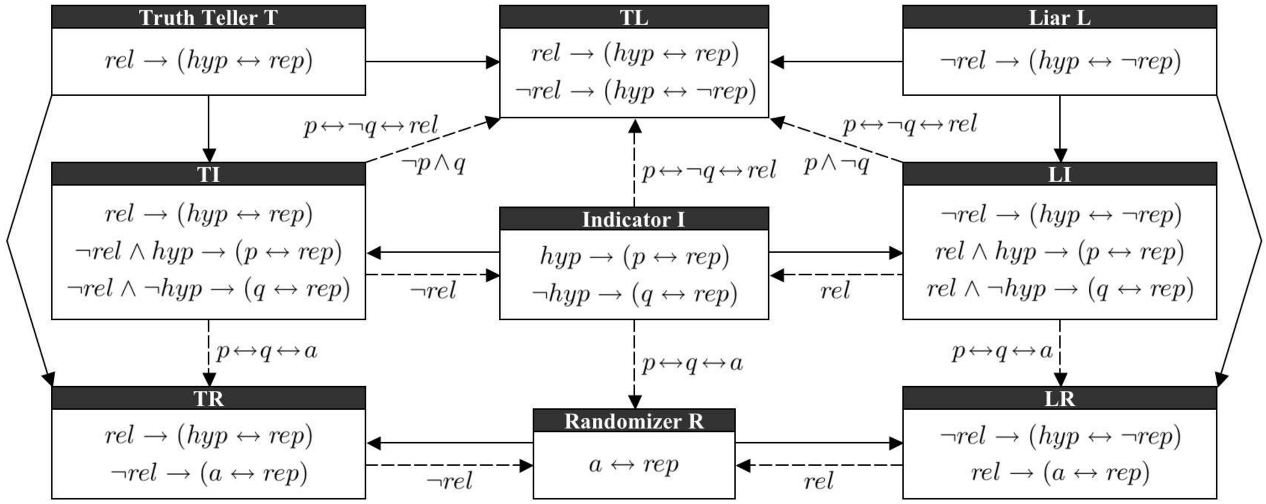

Fig. 1: Various models of partially reliable sources and their relationships.

the information source lies and the report is incorrect. This is expressed by

$$\neg rel \rightarrow (hyp \leftrightarrow \neg rep),$$

but nothing is said about the behavior of the information source in case of rel.

(TL) Consider the combination of the models (T) and (L). This leads to a model in which the behavior of the information source is fully specified:

$$rel \rightarrow (hyp \leftrightarrow rep),$$
$$\neg rel \rightarrow (hyp \leftrightarrow \neg rep).$$

Note that this corresponds to the third introductory model discussed in Section 1. It will lead to a formula equivalent to (3).

(I) *Indicator*: This is another model in which the outcome of the report is fully specified. It depends primarily on HYP and secondarily on randomization parameters p and q with corresponding probabilities:

$$hyp \rightarrow (p \leftrightarrow rep),$$
$$\neg hyp \rightarrow (q \leftrightarrow rep).$$

This model is equivalent to the first introductory model discussed in Section 1. It will confirm the result shown in (1).

(TI) Consider the combination of the models (T) and (I). This is model in which the outcome of the report depends primarily on REL, secondarily on HYP, and finally on randomization parameters p and q:

$$rel \rightarrow (hyp \leftrightarrow rep),$$
$$\neg rel \wedge hyp \rightarrow (p \leftrightarrow rep),$$
$$\neg rel \wedge \neg hyp \rightarrow (q \leftrightarrow rep).$$

(LI) Consider the combination of the models (L) and (I). With respect to REL, it is symmetric to model (TI):

$$\neg rel \rightarrow (hyp \leftrightarrow \neg rep),$$
$$rel \wedge hyp \rightarrow (p \leftrightarrow rep),$$
$$rel \wedge \neg hyp \rightarrow (q \leftrightarrow rep).$$

(R) *Randomizer*: This model describes a valueless information source producing random reports independently of HYP. Thus, the outcome of REP only depends uniquely on a randomization parameters a:

$$a \leftrightarrow rep.$$

Note that (R) is a special case of (I) for $p = q = a$.

(TR) Consider the combination of the models (T) and (R). The outcome of the report depends then primarily on REL and secondarily on the randomization parameter a:

$$rel \rightarrow (hyp \leftrightarrow rep),$$
$$\neg rel \rightarrow (a \leftrightarrow rep).$$

This model corresponds to the second introductory model discussed in Section 1 and it will confirm the result shown in (2).

(LR) Consider the combination of the models (L) and (R). With respect to REL, it is symmetric to model (TR):

$$\neg rel \rightarrow (hyp \leftrightarrow \neg rep),$$
$$rel \rightarrow (a \leftrightarrow rep).$$

Some of the above models are special cases of other models. For example, $\neg rel$ makes (I) a special case of (TI) and (R) a special case of (TR). Symmetrically, rel makes (I) a special case of (LI) and (R) a special case of (LT). Furthermore, $p \leftrightarrow q \leftrightarrow a$ makes (TR), (R), (LR) special cases of (TI), (I), and (LI), respectively. Finally, (TL) is a special case of (TI), (I), and (LI) for $p \leftrightarrow \neg q \leftrightarrow rel$ and of (TI) and (LI) for $\neg p \wedge q$ and $p \wedge \neg q$, respectively. Figure 1 illustrates the relationship between the various models. Dashed arrows indicate possible special cases. Note that (T), (TI), (L), and (LI) are the only models that are not special cases of others.

2.2 COMPUTING THE PARAMETERS

Now let's direct our attention to the determination of the parameters x_1, y_1, z_1 for positive reports and x_2, y_2, z_2 for negative reports. Note that the assumption-based descriptions of the various models lead to corresponding *hints* in the sense of [9]. In all cases, the *frame of discernment* is simply the Cartesian product

$$\begin{aligned}\Theta &= HYP \times REP \\ &= \{(hyp, rel), (hyp, \neg rel), (\neg hyp, rel), (\neg hyp, \neg rel)\}.\end{aligned}$$

The set of interpretations depends on the assumptions involved in the model. In the following, we will focus the discussion on the (TR) model. At the end, all other results will be provided by a theorem. The (TR) model includes two assumptions rel and a. The Cartesian product

$$\Omega = \{(rel, a), (rel, \neg a), (\neg rel, a), (\neg rel, \neg a)\}$$

forms thus the set of possible interpretations with $p(rel, a) = a\rho$, $p(rel, \neg a) = \bar{a}\rho$, $p(\neg rel, a) = a\bar{\rho}$, and $p(\neg rel, \neg a) = \bar{a}\bar{\rho}$. The corresponding consequences for Θ are

$$\begin{aligned}\Gamma(rel, a) = \Gamma(rel, \neg a) &= \{(hyp, rep), (\neg hyp, \neg rep)\}, \\ \Gamma(\neg rel, a) &= \{(hyp, rep), (\neg hyp, rep)\}, \\ \Gamma(\neg rel, \neg a) &= \{(hyp, \neg rep), (\neg hyp, \neg rep)\}.\end{aligned}$$

This defines a mass function m_{TR} over Θ. The only non-zero masses are

$$\begin{aligned}m_{\mathrm{TR}}(\{(hyp, rep), (\neg hyp, \neg rep)\}) &= \rho, \\ m_{\mathrm{TR}}(\{(hyp, rep), (\neg hyp, rep)\}) &= a\bar{\rho}, \\ m_{\mathrm{TR}}(\{(hyp, \neg rep), (\neg hyp, \neg rep)\}) &= \bar{a}\bar{\rho}.\end{aligned}$$

Now suppose that the information source provides a positive report. The above mass function has then to be conditioned on rep. The result is a new mass function m^{+}_{TR} over Θ with

$$\begin{aligned}m^{+}_{\mathrm{TR}}(\{(hyp, rep)\}) &= \rho, \\ m^{+}_{\mathrm{TR}}(\{(hyp, rep), (\neg hyp, rep)\}) &= a\bar{\rho}, \\ m^{+}_{\mathrm{TR}}(\emptyset) &= \bar{a}\bar{\rho}.\end{aligned}$$

After normalization and by projecting m^{+}_{TR} to $\{HYP\}$, we get the first three parameters for the (TR) model:

$$x_1 = \frac{a\bar{\rho}}{1 - \bar{a}\bar{\rho}}, \quad y_1 = 1, \quad z_1 = \frac{a\bar{\rho}}{1 - \bar{a}\bar{\rho}}.$$

Similarly, if the information source provides a negative report, we derive from m_{TR} a new mass function m^{-}_{TR} over Θ with

$$\begin{aligned}m^{-}_{\mathrm{TR}}(\{(\neg hyp, \neg rep)\}) &= \rho, \\ m^{-}_{\mathrm{TR}}(\emptyset) &= a\bar{\rho}, \\ m^{-}_{\mathrm{TR}}(\{(hyp, \neg rep), (\neg hyp, \neg rep)\}) &= \bar{a}\bar{\rho}.\end{aligned}$$

After normalization and by projecting m^{-}_{TR} to $\{HYP\}$, we get the remaining parameters for the (TR) model:

$$x_2 = 1, \quad y_2 = \frac{\bar{a}\bar{\rho}}{1 - a\bar{\rho}}, \quad z_2 = \frac{\bar{a}\bar{\rho}}{1 - a\bar{\rho}}.$$

A similar procedure is possible for all other models. For more information about the transformation from an assumption-based propositional language to hints and from hints to mass functions we refer to the corresponding literature [12].

Theorem 2 *The results shown in the following table are the parameters for the models introduced in the previous subsection.*

	Positive Report			*Negative Report*		
Model	x_1	y_1	z_1	x_2	y_2	z_2
(T)	$\bar{\rho}$	1	$\bar{\rho}$	1	$\bar{\rho}$	$\bar{\rho}$
(L)	1	ρ	ρ	ρ	1	$\bar{\rho}$
(TL)	$\bar{\rho}$	ρ	0	ρ	$\bar{\rho}$	0
(R)	1	1	1	1	1	1
(TR)	$\frac{a\bar{\rho}}{1-\bar{a}\bar{\rho}}$	1	$\frac{a\bar{\rho}}{1-\bar{a}\bar{\rho}}$	1	$\frac{\bar{a}\bar{\rho}}{1-a\bar{\rho}}$	$\frac{\bar{a}\bar{\rho}}{1-a\bar{\rho}}$
(LR)	1	$\frac{a\rho}{1-\bar{a}\rho}$	$\frac{a\rho}{1-\bar{a}\rho}$	$\frac{\bar{a}\rho}{1-a\rho}$	1	$\frac{\bar{a}\rho}{1-a\rho}$
(I)	$\frac{q}{1-\bar{p}\bar{q}}$	$\frac{p}{1-\bar{p}\bar{q}}$	$\frac{pq}{1-\bar{p}\bar{q}}$	$\frac{\bar{q}}{1-pq}$	$\frac{\bar{p}}{1-pq}$	$\frac{\bar{p}\bar{q}}{1-pq}$
(TI)	$\frac{q\bar{\rho}}{1-\bar{p}\bar{q}\bar{\rho}}$	$\frac{1-\bar{p}\bar{\rho}}{1-\bar{p}\bar{q}\bar{\rho}}$	$\frac{pq\bar{\rho}}{1-\bar{p}\bar{q}\bar{\rho}}$	$\frac{1-q\bar{\rho}}{1-pq\bar{\rho}}$	$\frac{\bar{p}\bar{\rho}}{1-pq\bar{\rho}}$	$\frac{\bar{p}\bar{q}\bar{\rho}}{1-pq\bar{\rho}}$
(LI)	$\frac{1-\bar{q}\bar{\rho}}{1-\bar{p}\bar{q}\rho}$	$\frac{p\rho}{1-\bar{p}\bar{q}\rho}$	$\frac{pq\rho}{1-\bar{p}\bar{q}\rho}$	$\frac{\bar{q}\rho}{1-pq\rho}$	$\frac{1-p\bar{\rho}}{1-pq\rho}$	$\frac{\bar{p}\bar{q}\rho}{1-pq\rho}$

For the proof of this theorem we refer to one of the authors' forthcoming publications.

In Section 3, we will discuss the most interesting models by implanting the corresponding parameters into (8) and (9).

2.3 MODELING PRIOR KNOWLEDGE

Prior knowledge, if available, is an important factor to be considered. It is usually the product of previous reports from partially reliable sources that are not further specified. Or it may simply reflect a personal opinion, feeling, or prejudice over HYP. We will not further take the origin of the prior knowledge into account. This means that HYP is the only variable affected. As a consequence, we can represent prior knowledge as a mass function over HYP specified by parameter x, y, and z as explained before.

We have shown in (8) and (9) how the influence of prior knowledge over HYP is determined by three parameters x, y, and z. And we have already discussed two special cases of $z = 0$ (a prior probability $p(hyp) = h$ is given) and $z = 1$ (no prior knowledge available). In order to get a proper interpretation of these parameters, an interpretation that is applicable to the general case $0 \leq z \leq 1$, consider the following model (P):

(P) Let there be a prior probability h for hyp in which one is more or less confident. The strength of the confidence is expressed by $p(con) = \gamma$. We can then look at hyp as an event which depends primarily on con and secondarily on a randomization parameter a with $p(a) = h$:

$$con \rightarrow (a \leftrightarrow hyp).$$

Note that nothing is said about hyp in case of $\neg con$.

This model corresponds to a situation in which the holder of the prior knowledge is not totally sure about the opinion or feeling he has about hyp. This leads to parameters

$$x = 1 - \gamma h, \quad y = 1 - \gamma \bar{h}, \quad z = 1 - \gamma, \tag{13}$$

that can be implanted into (8) and (9). Note that $\gamma = 0$ implies $x = y = z = 1$. This is a situation where no prior knowledge is available. As a consequence, (8) and (9) simplify to (11) and (12). On the other hand, if $\gamma = 1$, we have $x = \bar{h}$, $y = h$ and $z = 0$. This is a situation where a prior probability $p(hyp) = h$ is available. It allows to transform both (8) and (9) to (10).

A pleasant property of the above model is the possibility of varying the strength of the prior knowledge continuously between 0 and 1. This seems to reflect a natural property of real prior knowledge.

3 CASE STUDIES

The purpose of this section is to study certain models of partially reliable sources more deeply. We consider the models (T), (TL), (I), and (TR) to be the most interesting ones. Furthermore, model (P) will be used for the given prior knowledge. By (PT), (PTL), (PI), and (PTR) we denote the corresponding combined models. The corresponding solutions for degree of support and degree of possibility are obtained by substituting the parameters in (8) and (9) by the respective values shown in Theorem 2 and (13). But we do not further discuss the general case because it does not allow significant simplifications.

Our discussion will thus be restricted to the two extreme cases $\gamma = 0$ and $\gamma = 1$. The corresponding results can be derived from (10), (11), and (12). We will use $\delta = n - m$ to denote the gap between positive and negative reports and $N = n + m$ for the total number of reports. In many practical applications one would expect the results to depend on both δ and N. For example, 10 positive and 0 negative reports is obviously a different situation than 1010 positive and 1000 negative reports.

3.1 Model (PT)

Consider the (T) model in which the outcome of REP only depends on REL. For $\gamma = 0$ (no prior knowledge), we can transform (11) and (12) into

$$dsp(hyp) \;=\; 1 - \frac{1}{1 + \bar{\rho}^{\,m-n} - \bar{\rho}^{\,m}}\,, \tag{14}$$

and

$$dps(hyp) \;=\; \frac{1}{1 + \bar{\rho}^{\,n-m} - \bar{\rho}^{\,n}}\,. \tag{15}$$

This seems to be a legitimate result, because for $\rho > 0$ both $dsp(hyp)$ and $dps(hyp)$ tend toward 1 for $n \to \infty$ and toward 0 for $m \to \infty$. Note that this is true for any $\rho > 0$. Furthermore, the results depend on both δ and N. If δ is fixed, $\rho > 0$, and $N \to \infty$, then

$$dsp(hyp) \;=\; dps(hyp) \;=\; \frac{1}{1 + \bar{\rho}^{\,\delta}}\,. \tag{16}$$

Now look at the case $\gamma = 1$ where a prior probability h is given. This allows to derive

$$dsp(hyp) \;=\; dps(hyp) \;=\; \frac{h}{h + \bar{h}\bar{\rho}^{\,n-m}} \tag{17}$$

from (11). The result tends again toward 1 and 0 for $n \to \infty$ and $m \to \infty$, respectively. Note that (17) is only a function of h, ρ, and δ, but not of N.

3.2 Model (PTL)

Consider the (TL) model in which it is distinguished between truth telling and lying. As a consequence $z_1 = z_2 = 0$, we get $dsp(hyp) = dps(hyp)$ independently of γ. If prior knowledge is not available, that is $\gamma = 0$, it follows from (11) and (12) that

$$dsp(hyp) \;=\; dps(hyp) \;=\; \frac{1}{1 + \left(\frac{\bar{\rho}}{\rho}\right)^{n-m}}\,. \tag{18}$$

For $m = 0$, this result includes Laplace's formula (3) as a special case [4]. Note that $\rho = \frac{1}{2}$ implies $dsp(hyp) = dps(hyp) = \frac{1}{2}$. Otherwise, if there are infinitely many positive reports, $n \to \infty$, we get

$$dsp(hyp) \;=\; dps(hyp) \;=\begin{cases} 1, \text{ if } \rho > \frac{1}{2}\,, \\ 0, \text{ if } \rho < \frac{1}{2}\,. \end{cases} \tag{19}$$

On the other hand, if there are infinitely many negative reports, $m \to \infty$, we get

$$dsp(hyp) \;=\; dps(hyp) \;=\begin{cases} 0, \text{ if } \rho > \frac{1}{2}\,, \\ 1, \text{ if } \rho < \frac{1}{2}\,. \end{cases} \tag{20}$$

Note that (19) and (20) is essentially the *Condorcet Jury Theorem* discussed in social choice theory [5]. The problem with (18), (19), and (20) is that the results only depend on ρ and δ, but not on N.

Now consider case of $\gamma = 1$ with a given prior probability $p(hyp) = h$. This allows to transform (10) into

$$dsp(hyp) \;=\; dps(hyp) \;=\; \frac{h}{h + \bar{h}\left(\frac{\bar{\rho}}{\rho}\right)^{n-m}}\,. \tag{21}$$

This result is similar to the one given in (18). It clearly demonstrates the impact of the prior probability h. The formula corresponds to the one given by Boole [6]. The parameters on which (21) depends are h, ρ, and δ, but not N.

3.3 Model (PI)

This is the model that corresponds to the first introductory model in Section 1. If there is no prior knowledge, that is if $\gamma = 0$, we can transform (11) and (12) into

$$dsp(hyp) \;=\; 1 - \frac{1}{1 + \left(\frac{p}{q}\right)^{n}\left(\frac{\bar{p}}{\bar{q}}\right)^{m} - p^n \bar{p}^m} \tag{22}$$

and

$$dps(hyp) \;=\; \frac{1}{1 + \left(\frac{q}{p}\right)^{n}\left(\frac{\bar{q}}{\bar{p}}\right)^{m} - q^n \bar{q}^m}\,. \tag{23}$$

Suppose that both p and q are strictly between 0 and 1. First, consider the special case of $p = q$. This allows to write (22) as

$$dsp(hyp) \;=\; 1 - \frac{1}{2 - p^n \bar{p}^m} \;=\; 1 - dps(hyp)\,. \qquad (24)$$

This result tends toward $\frac{1}{2}$ for both $n \to \infty$ or $m \to \infty$. Such an information source is valueless. Second, let p be different from q. Then $n \to \infty$ implies

$$dsp(hyp) \;=\; dps(hyp) \;=\begin{cases} 1, \text{ if } p > q\,, \\ 0, \text{ if } p < q\,. \end{cases} \qquad (25)$$

Similarly, $m \to \infty$ implies

$$dsp(hyp) \;=\; dps(hyp) \;=\begin{cases} 0, \text{ if } p > q\,, \\ 1, \text{ if } p < q\,. \end{cases} \qquad (26)$$

Thus, if the (PI) model is used to describe medical tests, for example, parameter p is expected to exceed q.

Finally, consider the case of $\gamma = 1$ and $p(hyp) = h$. By implanting the corresponding parameters into (10), we obtain

$$dsp(hyp) \;=\; dps(hyp) \;=\; \frac{h}{h + \bar{h}\big(\frac{q}{p}\big)^n \big(\frac{\bar{q}}{\bar{p}}\big)^m}\,. \qquad (27)$$

This result corresponds to formula (1) obtained in Section 1 with the aid of Bayesian networks [3]. We get $dsp(hyp) = dps(hyp) = h$ for $p = q$. If p is different from q and $0 < h < 1$, the limits for $n \to \infty$ and $m \to \infty$ are equal to (25) and (26), respectively.

3.4 Model (PTR)

The last model we analyze here is the one that corresponds to the second introductory example in Section 1. We suppose $0 < \rho < 1$ and $0 < a < 1$ throughout the discussion ($\rho = 0$ leads to the (R) model and $\rho = 1$ produces conflicts whenever both $n \geq 1$ and $m \geq 1$).

We start again with the case $\gamma = 0$ of no prior knowledge. This allows to write (11) and (12) as

$$dsp(hyp) \;=\; 1 - \frac{1}{1 + \big(\frac{\bar{a}\bar{\rho}}{1-a\bar{\rho}}\big)^m \big[\big(\frac{1-\bar{a}\bar{\rho}}{a\bar{\rho}}\big)^n - 1\big]} \qquad (28)$$

and

$$dps(hyp) \;=\; \frac{1}{1 + \big(\frac{a\bar{\rho}}{1-\bar{a}\bar{\rho}}\big)^n \big[\big(\frac{1-a\bar{\rho}}{\bar{a}\bar{\rho}}\big)^m - 1\big]}\,, \qquad (29)$$

respectively. Furthermore, we obtain $dsp(hyp) = dps(hyp) = 1$ for $n \to \infty$ and $dsp(hyp) = dps(hyp) = 0$ for $m \to \infty$. Finally, if δ is fix and $N \to \infty$, then the result depends on a:

$$dsp(hyp) \;=\; dps(hyp) \;=\begin{cases} 1, \text{ if } a > \frac{1}{2}\,, \\ \frac{1}{1+\left(\frac{1-\rho}{1+\rho}\right)^{n-m}}, \text{ if } a = \frac{1}{2}\,, \\ 0, \text{ if } a < \frac{1}{2}\,. \end{cases} \qquad (30)$$

To conclude this section, look at the case of $\gamma = 1$ and $p(hyp) = h$. We can then derive

$$dsp(hyp) \;=\; dps(hyp) \;=\; \frac{h}{h + \bar{h}\big(\frac{a\bar{\rho}}{1-\bar{a}\bar{\rho}}\big)^n \big(\frac{1-a\bar{\rho}}{\bar{a}\bar{\rho}}\big)^m} \qquad (31)$$

from (10). This formula corresponds to the one obtained in Section 1 with the aid of Bayesian networks. Provided that $0 < h < 1$, both $dsp(hyp)$ and $dps(hyp)$ tend toward 1 for $n \to \infty$ and toward 0 for $m \to \infty$. For fixed δ, it follows from $N \to \infty$ that

$$dsp(hyp) \;=\; dps(hyp) \;=\begin{cases} 1, \text{ if } a > \frac{1}{2}\,, \\ \frac{h}{h+\bar{h}\left(\frac{1-\rho}{1+\rho}\right)^{n-m}}, \text{ if } a = \frac{1}{2}\,, \\ 0, \text{ if } a < \frac{1}{2}\,. \end{cases} \qquad (32)$$

Note that this result only depends on ρ for $a = \frac{1}{2}$.

4 CONCLUSION

This paper approaches the problem of independent and partially reliable information sources from a very general perspective with Dempster-Shafer's theory of evidence as the underlying mathematical mechanism. The result is a generic model with a number of possible instantiations. The paper illuminates the relationship between the various instantiations and analyses corresponding conclusions. It also discusses the role of prior knowledge and proposes a model in which the influence of a given prior probability is controlled by a continuous confidence parameter γ.

There are a number of open questions. One of them concerns the problem of choosing the "right" model. We do not exclude the possibility of arguing in favor or against certain models, but we can't and we don't want to give a definite answer here. In our view, the choice of the model crucially depends on the circumstances of the concrete problem and the type of the available information. In this sense, we think that all models are legitimate.

Another open question is the treatment of dependencies. Relaxing the assumption of independent sources would certainly make the analysis more complicated, but it would also become more realistic. We postpone this important topic to future publications.

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